Th.: The union of a countable family of pairwise disjoint countable sets is countable.

Proof:- Let $\langle A_n \rangle$, $n = 1, 2, 3, \ldots$ be sequence of pairwise disjoint countable sets or countable family of countable sets.
To prove that $\bigcup_{n=1}^{\infty} A_n$ is countable.

We can write the elements of A_n in the form of arrays, as follows.

$$
\begin{array}{lllllll}
A_1 : & a_{11} & & a_{12} & \rightarrow & a_{13} & a_{14} \; \cdots \\
 & \downarrow & & \uparrow & & \downarrow & \\
A_2 : & a_{21} & \rightarrow & a_{22} & & a_{23} & \cdots\cdots \\
 & & & & & \downarrow & \\
A_3 : & a_{31} & \leftarrow & a_{32} & \leftarrow & a_{33} & a_{34} \; \cdots \\
 & \downarrow & & & & & \\
- & - & - & - & - & - & - \\
- & - & - & - & - & - & -
\end{array}
$$

⇒ Now, we shall arrange all the elements of $A = \bigcup_{n=1}^{\infty} A_n$ in the form of a sequence, then we can display | arrange these elements as follows.

Ist Block : a_{11} ; $i+j=2$

IInd Block : a_{21} a_{12} $i+j=3$

IIIrd Block : a_{31} a_{22} a_{13} $i+j=4$

- - - - - - - - - - - - - -

mth Block a_{m1} $a_{m-1\,2}$ $a_{m-2\,3}$ - - - $i+j=m+1$

- - - - - - - - - - - - - -

$A = \{ \underline{a_{11}}; \underline{a_{21}, a_{12}}; \underline{a_{31}, a_{22}, a_{13}}; \ldots \}$

⇒ A is countable. H.P.

Properties of Non-denumerable sets

Th. The set of all real numbers in $[0, 1]$ is non-denumerable or uncountable.

Proof:- Let us assume that $[0,1]$ is countable if possible

⇒ The members of $[0,1]$ can be written in the form of a sequence.

i.e. like $\langle a_1, a_2, a_3, \ldots \rangle$

where each $a_i \in [0,1]$ $\forall\, i \in \mathbb{N}$. (8)

Therefore, we can expand each a_i in the form of infinite decimals as follows

$a_1 = 0.a_{11}\, a_{12}\, a_{13}\, a_{14} \cdots$ 0.4593---

$a_2 = 0.a_{21}\, a_{22}\, a_{23}\, a_{24} \cdots$ 0.3849---

- - - - - - - - - -

- - - - - - - - - -

$a_n = 0.a_{n1}\, a_{n2}\, a_{n3}\, a_{n4} \cdots$

- - - - - - - - - -

- - - - - - - - - -

where $a_{ij} \in [0,1,2,3,\ldots,9]$ and each decimal representation contains an infinite number of non zero elements

Here we write 1 as $0.99999---$ ✓

like $\frac{1}{2} = .5$ as $0.49999---$ ✓

0.50000 — ✗

Now we construct a real number $y = 0.\cancel{a}\, b_1 b_2 b_3 b_4 \cdots$ which belongs to $[0,1]$ in the following way.

choose b_1 in such a way that $b_1 \neq a_{11}$

———— b_2 ———————— $b_2 \neq a_{22}$

———— b_3 ———————— $b_3 \neq a_{33}$

– – — b_n ———————— $b_n \neq a_{nn}$

etc. — — — —

By doing so, we can say that

$y \neq a_1$, $y \neq a_2$, $\cdots$ $y \neq a_n$ - - - - -

$\Rightarrow$ y is not amongst the numbers $\{a_1, a_2, a_3 \cdots a_n \cdots\}$

$\Rightarrow$ y is different than all a_is and these a_is form the elements of $[0,1]$

$\Rightarrow$ we have constructed a number y by choosing this technique,

This y is a member of $[0,1]$ because it is of the form of $0.b_1 b_2 b_3 \ldots$ and $\notin [0,1]$

But this is a contradiction to our assumption. Because it must be amongst $\langle a_1, a_2 \ldots \rangle$

$\Rightarrow$ elements | members of $[0,1]$ Cann't be ~~expressed~~ arranged in the form of a sequence.

$\Rightarrow$ $[0,1]$ is uncountable

$\Rightarrow$

Non denumerable set